

Sign conjugacy classes of the alternating groups

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Abstract

A conjugacy class C of a finite group G is a sign conjugacy class if every irreducible character of G takes value 0, 1 or -1 on C . In this paper we classify the sign conjugacy classes of alternating groups.

1 Introduction

We will begin this paper by giving the definition of sign conjugacy class for an arbitrary finite group.

Definition 1.1. *Let G be a finite group. A conjugacy class of G is a sign conjugacy class of G if every irreducible character of G takes values 0, 1 or -1 on C .*

In [4] Olsson considered sign conjugacy classes of S_n in order to answer a question from Isaacs and Navarro for S_n , using the following property of sign conjugacy classes:

Theorem 1.2. *Let C be a sign conjugacy class of a finite group G and define χ^+ and χ^- by*

$$\chi^+ := \sum_{\substack{\chi \in \text{Irr}(G): \\ \chi(C) > 0}} \chi \quad \text{and} \quad \chi^- := \sum_{\substack{\chi \in \text{Irr}(G): \\ \chi(C) < 0}} \chi.$$

Then χ^+ and χ^- are characters of G differing only on C .

In [4] Olsson also formulated a conjecture about sign conjugacy classes of S_n which was proven in [2]. In this paper we will classify sign conjugacy classes of A_n , proving that they are closely related to those of S_n , as would be to expect due to the relationship of irreducible characters of S_n and A_n .

As we will be working with alternating (and symmetric) groups, we will refer to partitions instead of conjugacy classes. This leads to the following two definitions:

Definition 1.3. *A partition γ is a sign partition if it is the cycle partition of a sign conjugacy class of S_n .*

If γ is an even partition then γ is an A_n -sign partition if it is the cycle partition of a sign conjugacy class of A_n .

For example (3) is a sign partition which is not an A_n -sign partition, while $(2, 2)$ is an A_n -sign partition which is not a sign partition. In general however A_n -sign partitions are almost always sign partitions and almost all even sign partitions are A_n sign partitions, as will be seen from Theorem 1.6 and Theorem 1.3 of [2].

For γ consisting of odd distinct parts it is clear that $C_{\gamma,+}$ is a sign conjugacy class of A_n if and only if $C_{\gamma,-}$ is also such a conjugacy class (later it will be proved that no such conjugacy class is a sign conjugacy class). In particular any conjugacy class of A_n is a sign conjugacy class exactly when its cycle partition is an A_n -sign partition.

In order to describe A_n -sign partitions we will need the two following sets of partitions:

Definition 1.4. *We define Sign to be the subsets of partitions consisting of all partitions $(\gamma_1, \dots, \gamma_r)$ for which there exists an s , $0 \leq s \leq r$ such that the following holds:*

- $\gamma_i > \gamma_{i+1} + \dots + \gamma_r$ for $1 \leq i \leq s$,
- $(\gamma_{s+1}, \dots, \gamma_r)$ is one of the following partitions:
 - $()$, $(1, 1)$, $(3, 2, 1, 1)$ or $(5, 3, 2, 1)$,
 - $(a, a-1, 1)$ with $a \geq 2$,
 - $(a, a-1, 2, 1)$ with $a \geq 4$,
 - $(a, a-1, 3, 1)$ with $a \geq 5$.

Definition 1.5. *We define $\overline{\text{Sign}}$ by:*

$$\begin{aligned} \overline{\text{Sign}} := & \{(1, 1, 1), (2, 2), (2, 2, 1), (5, 4, 3, 2, 1)\} \\ & \cup \{(a, a-1, 4, 1) : a \geq 6\} \\ & \cup \{(a, a-3, 2, 1, 1) : a = 6 \text{ or } a \geq 8\} \\ & \cup \{(a, a-5, 3, 2, 1) : 9 \leq a \leq 10 \text{ or } a \geq 12\} \\ & \cup \{(a, b, a-b+1) : b+1 \leq a \leq 2b-2\} \\ & \cup \{(a, b, a-b-1, 1) : b+2 \leq a \leq 2b\}. \end{aligned}$$

In [2] (Theorem 1.3) it was proven that a partition γ is a sign partition if and only if $\gamma \in \text{Sign}$, proving a conjecture of Olsson from [4]. In this paper we will prove the following characterization of A_n -sign conjugacy classes.

Theorem 1.6. *Let γ be a partition of $n \geq 2$. Then γ is an A_n -sign partition if and only if*

$$\gamma \in (\text{Sign} \cup \overline{\text{Sign}}) \cap \{\text{even partitions not consisting of odd distinct parts}\}.$$

In order to prove Theorem 1.6 we will use the following lemmas.

Lemma 1.7. *Let γ be an A_n -sign partition. If $\gamma \notin \{(1, 1, 1), (2, 2), (2, 2, 1)\}$ then γ does not have repeated parts, except possibly for the part 1, which may have multiplicity 2.*

Lemma 1.8. *Let $\alpha = (\alpha_1, \dots, \alpha_h)$ be a partition with $h \geq 3$. Assume that $\alpha_1 > \alpha_2$, that $\alpha \notin \text{Sign}$ and that $(\alpha_2, \dots, \alpha_h) \in \text{Sign}$. Then if $\alpha \neq (5, 4, 3, 2, 1)$ we can find a partition β of $|\alpha|$ such that $\chi_\alpha^\beta \notin \{0, \pm 1\}$ and $h_{2,1}^\beta = \alpha_1$.*

If $\alpha \notin \overline{\text{Sign}}$ we can choose β to be not self conjugate.

These two lemmas will allow us to prove in Section 3 that, if $\gamma = (\gamma_1, \dots, \gamma_r)$ is an A_n -sign partition, then $(\gamma_i, \dots, \gamma_r) \in \text{Sign}$ for $i \geq 2$ and $\gamma \in \text{Sign} \cup \overline{\text{Sign}}$, by otherwise constructing partitions δ which are not self conjugate and which satisfy $\chi_\gamma^\delta \notin \{0, \pm 1\}$, contradicting then the assumption that γ is an A_n -partition. In Section 4 we will prove that partitions consisting of odd distinct parts are not A_n -sign partitions (by simply looking at the characters of A_n indexed by the corresponding self conjugate partition), proving then one direction of Theorem 1.6. The other direction will be proved in Section 5. Through all of the following γ will be a partition of $n \geq 2$.

From Theorem 1.3 of [2] and from Theorem 1.6 we also easily obtain that if $\chi_\gamma^\delta \notin \{0, \pm 1\}$ and $\gamma \in \overline{\text{Sign}}$ does not consist of distinct odd parts, then δ is self conjugate. This can be proved also if $\gamma \in \overline{\text{Sign}}$ consist of distinct odd parts, using arguments similar to those from Section 5.

Proofs of results about irreducible characters of S_n and A_n and about weights, cores and quotients of partitions can be found in [1] and [3].

2 Proof of Lemmas 1.7 and 1.8

We will now prove Lemmas 1.7 and 1.8. To do this we will use results from [2] and [4]. Since most of the partitions considered there are of the form $(a, b, 1^c)$ we will first classify in the next lemma which such partitions are self conjugate. In the following if β is a partition we will write β' for its conjugate.

Lemma 2.1. *A partition $(a, b, 1^c)$ of $n \geq 2$ is self conjugate if and only if $c = a - 2$ and $b \in \{1, 2\}$. In particular if $(a, b, 1^c)$ is self conjugate then $b + c \in \{a - 1, a\}$.*

Proof. Let $\beta := (a, b, 1^c)$. If $b = 0$ clearly β is not self conjugate, as $a = n \geq 2$. So assume that $b \geq 1$. From $\beta'_2 \leq 2$ it follows that if β is self conjugate then $b \in \{1, 2\}$. As clearly $(a, 1^{c+1})$ and $(a, 2, 1^c)$ are self conjugate if and only if $c = a - 2$ the lemma follows. \square

We will now prove Lemma 1.7.

Proof of Lemma 1.7. Assume that γ has a repeated part (with multiplicity at least 3 if this part is 1).

If $\gamma \notin \{(1, 1, 1), (2, 2)\}$ and γ is not of the form $(a, a, 1)$ for $a \geq 4$ or of the form $(\delta_1, \dots, \delta_t, \mu_1, \dots, \mu_v)$ with $\delta_t > \mu_1$ and

$$\mu \in \{(2^m, 1), (2^m, 1^2), (3^m, 2, 1), (3^m, 2, 1^2), (3^m, 1), (3^m, 1^2) : m \geq 2\},$$

we can find from the proof of Lemma 6 and Theorem 7 of [4] a partition β with $\chi_\gamma^\beta \neq \{0, \pm 1\}$ which, from Lemma 2.1, is not self conjugate. It then easily follows that γ is not an A_n -sign partition. In the not covered cases let

$$\beta = \begin{cases} (a+1, a), & \gamma = (a, a, 1), \\ (n-2, 1^2), & \mu = (2^m, 1) \text{ or } (2^m, 1^2), \\ (n-3, 3), & \mu = (3^m, 2, 1), (3^m, 2, 1^2) \text{ or } (3^m, 1), \\ (n-3, 1^3), & \mu = (3^m, 1^2), \end{cases}$$

where $n = |\gamma|$. It can be easily checked using Lemma 2.1 that the above partitions are not self-adjoint unless $\gamma = (2, 2, 1)$. Also $\chi_\gamma^\beta \notin \{0, \pm 1\}$, for example

$$\chi_{(\delta_1, \dots, \delta_t, 2^m, 1)}^{(n-2, 1^2)} = \chi_{(2^m, 1)}^{(2m-1, 1^2)} = -(m-1)\chi_{(2, 1)}^{(3)} + \chi_{(2, 1)}^{(1^3)} = -m.$$

The lemma then follows. \square

We will now prove Lemma 1.8. Most of the work will be in proving that if $\alpha \notin \overline{\text{Sign}}$ then the partitions β constructed in the proof of Theorem 1.6 of [2] are not self-adjoint. This is always the case apart for $\alpha = (\alpha_2 + 2a - 1, \alpha_2, a, a - 1, 1)$ with $\alpha_2 > 2a$ and $a \geq 4$, which will be treated separately.

Proof of Lemma 1.8. We will divide the proof of the lemma in the following cases: 1) $\alpha = (\alpha_2 + 2a - 1, \alpha_2, a, a - 1, 1)$ with $\alpha_2 > 2a$ and $a \geq 4$, 2) all other cases. Case 2) will be divided in subcases corresponding to the different cases of the proof of Theorem 1.6 of [2].

1) For $\alpha = (\alpha_2 + 2a - 1, \alpha_2, a, a - 1, 1)$ with $\alpha_2 > 2a$ and $a \geq 4$ let $\beta := (\alpha_2 + 2a, 3, 1^{\alpha_2 + 2a - 4})$. Since $a \geq 4$ we have that β is a partition. Also $h_{2,1}^\beta = \alpha_2 + 2a - 1 = \alpha_1$ and β is not self-adjoint from Lemma 2.1. As

$$h_{1,3}^\beta = \alpha_2 + 2a + 2 - 3 = \alpha_1,$$

as $a \geq 4$ so that

$$\alpha_2 + 2a - 4 > \alpha_2$$

and as any partition of $\alpha_2 + 2a$ has at most one α_2 -hooks since $\alpha_2 > 2a$, we have

$$\begin{aligned} \chi_\alpha^\beta &= (-1)^{\alpha_2+2a-4} \chi_{(\alpha_2, a, a-1, 1)}^{(\alpha_2+2a)} - \chi_{(\alpha_2, a, a-1, 1)}^{(2, 2, 1^{\alpha_2+2a-4})} \\ &= (-1)^{\alpha_2+2a-4} - (-1)^{\alpha_2-1} \chi_{(a, a-1, 1)}^{(2, 2, 1^{2a-4})} \\ &= (-1)^{\alpha_2+2a-4} - (-1)^{\alpha_2-1+a-1} \chi_{(a-1, 1)}^{(2, 2, 1^{a-4})} \\ &= (-1)^{\alpha_2+2a-4} - (-1)^{\alpha_2-1+a-1+a-3} \chi_{(1)}^{(1)} \\ &= (-1)^{\alpha_2} 2. \end{aligned}$$

So the lemma holds in this case.

2) In each of the following cases β is as constructed in [2] for the corresponding case. Notation used here is as in [2].

- For

$$\begin{aligned} (\alpha_2, \dots, \alpha_h) &\in \{(1, 1), (3, 2, 1, 1), (5, 3, 2, 1)\} \\ &\quad \cup \{(a, a-1, 1) : 2 \leq a \leq 4\} \\ &\quad \cup \{(a, a-1, 2, 1) : 4 \leq a \leq 8\} \\ &\quad \cup \{(a, a-1, 3, 1) : 5 \leq a \leq 10\} \end{aligned}$$

the lemma can be checked by looking at each single case separately (they are finitely many since $\alpha_1 \leq \alpha_2 + \dots + \alpha_h$).

- For

$$\begin{aligned} (\alpha_2, \dots, \alpha_h) &\in \{(a, a-1, 1) : a \geq 5\} \\ &\quad \cup \{(a, a-1, 2, 1) : a \geq 9\} \\ &\quad \cup \{(a, a-1, 3, 1) : a \geq 11\} \end{aligned}$$

we can apply Lemma 2.1 to results from Section 2 of [2] and, as $\alpha_1 > \alpha_2 = a \geq 5$, obtain that if β is self conjugate then α is either $(2a, a, a-1, 1)$ or $(a+1, a, a-1, 1)$, as in all other cases $\beta = (|\alpha| - \alpha_1, b, 1^{\alpha_1-b})$ with $|\alpha| - \alpha_1 \geq \alpha_1 + 2$ or $b \geq 4$.

If $\alpha = (a+1, a, a-1, 1)$ then $\beta = (a-1, a-1, a-1, 4)$ is self conjugate if and only if $a = 5$, that is $\alpha = (6, 5, 4, 1) \in \overline{\text{Sign}}$.

If $\alpha = (2a, a, a-1, 1) = (2a, a, 2a-a-1, 1)$. As $a \geq 5$ we have that $\alpha \in \overline{\text{Sign}}$.

- For α as in Theorems 3.1 or 3.2 of [2] we have from Lemma 2.1 that if β is self conjugate then $x = 1$ and $|\alpha| - \alpha_1 = \alpha_1$. From $x = 1$ we have that $k = h$ and $\alpha_h = 1$. From $|\alpha| - \alpha_1 = \alpha_1$ we have that $\alpha_1 = \alpha_2 + \dots + \alpha_h$, that is $\alpha_1 - \alpha_2 = \alpha_3 + \dots + \alpha_h$, so that $k = 4$. So $\alpha = (\alpha_1, \alpha_2, \alpha_1 - \alpha_2 - 1, 1)$. From $\alpha_1 - \alpha_2 > \alpha_h = 1$ and $\alpha_2 > \alpha_1 - \alpha_2 - 1$ it follows that $\alpha \in \overline{\text{Sign}}$ when β is self-adjoint.

- For α as in Theorem 3.3 of [2], β is not self-adjoint from Lemma 2.1, as

$$\beta_2 = \alpha_1 - c = \alpha_2 + \alpha_{k-1} + \dots + \alpha_h - c > \alpha_2 > 2.$$

- For α as in Theorem 3.4 of [2] we have that if β is self conjugate then $k = 4$ and $(\alpha_{k-1}, \dots, \alpha_h) = (a, a - 1, 1)$, as in all other cases $\alpha_1 \neq |\alpha| - \alpha_1 - 1$. Here $\alpha_1 = \alpha_2 + 2a - 1$ and $\alpha_2 > 2a$. The case $a \geq 4$ has already been considered in 1). For $a = 2$ or $a = 3$ instead we have $\alpha \in \overline{\text{Sign}}$.

- For α as in Theorems 3.5, 3.9 and 3.11 of [2], β is not self-adjoint from Lemma 2.1, as $\alpha_1 > 2$.

- For α as in Theorems 3.6, 3.7 and 3.10 of [2] the lemma follows again from Lemma 2.1, as $\alpha_2 > 1$.

- For α as in Theorem 3.8 of [2], if β is self-adjoint then $\alpha_1 = |\alpha| - \alpha_1 + 1$. By assumption $|\alpha| - \alpha_1 \geq \alpha_2 + \alpha_h > \alpha_1$, so that if β is self-adjoint then $h = 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_1 - \alpha_2 + 1)$. As $\alpha_1 > \alpha_2 > \alpha_1 - \alpha_2 + 1$, so that $\alpha_2 + 1 \leq \alpha_1 \leq 2\alpha_2 - 2$, it follows that $\alpha \in \overline{\text{Sign}}$ when β is not self conjugate.

- For α as in Theorem 3.12 of [2] we have that $\beta_2 = \alpha_3 \geq 4 = \beta'_2$. In particular β is not self conjugate for $\alpha_3 \geq 5$. For $\alpha_3 = 4$ then $\alpha = (\alpha_1, \alpha_1 - 1, 4, 1) \in \overline{\text{Sign}}$, as $\alpha_1 - 1 > 4$, so that $\alpha_1 \geq 6$.

- For α as in Theorem 3.13 of [2] we have that $\alpha_3 > \alpha_4 \geq \alpha_{h-1} = 2$ as $h \geq 5$ and then $\alpha_1 > \alpha_2 > \alpha_3 + \alpha_4 \geq 5$. In particular $\alpha_1 \geq 7$ and then $\beta'_2 = 4 < \alpha_1 - 2 = \beta_2$ so that β is not self conjugate.

- For α as in Theorem 3.14 of [2] if β is self-adjoint then $\alpha_{h-1} = 3$ and

$$|\alpha| - \alpha_1 - \alpha_{h-1} + 1 = \alpha_1 - \alpha_{h-1} - 1 + \alpha_{h-1} - 3 + 3.$$

So $\alpha_2 + \dots + \alpha_h = \alpha_1 + 1$, which from the assumptions is equivalent to $\alpha_3 + \dots + \alpha_{h-1} = 1$. Always from the assumption this would give

$$3 < \alpha_3 + \dots + \alpha_{h-1} = 1$$

leading to a contradiction. So β is not self-adjoint.

□

3 A_n -sign partitions are elements of $\text{Sign} \cup \overline{\text{Sign}}$

Let γ be an A_n -sign partition. If $r \leq 2$ then clearly $\gamma \in \text{Sign} \cup \overline{\text{Sign}}$ from Lemma 1.7 (if $r = 2$ and $\gamma_1 = \gamma_2$ then $\gamma \in \{(1, 1), (2, 2)\}$).

So assume now that $r \geq 3$. From Lemma 1.7 we have that $(\gamma_{r-1}, \gamma_r) \in \text{Sign}$. Also either $\gamma \in \{(1, 1, 1), (2, 2, 1)\}$ and then $\gamma \in \overline{\text{Sign}}$ or $\gamma_i > \gamma_{i+1}$ for $1 \leq i \leq r-2$. Assume now that for some $2 \leq i \leq r-2$ we have $(\gamma_{i+1}, \dots, \gamma_r) \in \text{Sign}$ and $(\gamma_i, \dots, \gamma_r) \notin \text{Sign}$. Then $\gamma_i > \gamma_{i+1}$ and so, if $(\gamma_i, \dots, \gamma_r) \neq (5, 4, 3, 2, 1)$, we can find β with $\chi_{(\gamma_i, \dots, \gamma_r)}^\beta \notin \{0, \pm 1\}$ and $h_{2,1}^\beta = \gamma_i$. From $2 \leq i \leq r-2$ it follows $r \geq 4$, so that $\gamma \notin \{(1, 1, 1), (2, 2, 1)\}$ and then $\gamma_j > \gamma_i$ for $j < i$. Let

$$\delta := (\beta_1 + \gamma + 1 + \dots + \gamma_{i-1}, \beta_2, \beta_3, \dots).$$

As $h_{2,1}^\delta = h_{2,1}^\beta = \gamma_i$ and $i \geq 2$ we have

$$\delta'_1 \leq h_{2,1}^\delta + 1 = \gamma_i + 1 < 1 + \gamma_1 \leq \delta_1,$$

so that δ is not self conjugate. From

$$\chi_\gamma^\delta = \chi_{(\gamma_i, \dots, \gamma_r)}^\beta \notin \{0, \pm 1\},$$

we then have a contradiction to γ being an A_n -sign partition.

Assume now that $(\gamma_i, \dots, \gamma_r) = (5, 4, 3, 2, 1)$. If $\gamma_{i-1} \geq 7$ let

$$\delta := (4 + \gamma_1 + \dots + \gamma_{i-1}, 4, 4, 3).$$

In this case $\delta'_1 = 4 < 11 \leq \delta_1$, so that also in this case δ is not self conjugate.

As $\gamma_{i-1} \geq 7$,

$$\chi_\gamma^\delta = \chi_{(5,4,3,2,1)}^{(4,4,4,3)} = -2$$

and then also in this case we have a contradiction. If instead $\gamma_{i-1} = 6$ let

$$\delta := (15 + \gamma_1 + \dots + \gamma_{i-2}, 2, 1, 1, 1, 1).$$

Here too δ is not self conjugate and

$$\chi_\gamma^\delta = \chi_{(6,5,4,3,2,1)}^{(15,2,1,1,1,1)} = 2,$$

again leading to a contradiction.

By induction $(\gamma_2, \dots, \gamma_r) \in \text{Sign}$. Assume now that $\gamma \notin \text{Sign} \cup \overline{\text{Sign}}$. Then from Lemma 1.8 (as $\gamma \notin \{(1, 1, 1), (2, 2, 1)\}$, so that $\gamma_1 > \gamma_2$ from Lemma 1.7 in this case) there exists β not self conjugate with $\chi_\gamma^\beta \notin \{0, \pm 1\}$, again leading to a contradiction.

In particular if γ is an A_n -sign partition then $\gamma \in \text{Sign} \cup \overline{\text{Sign}}$.

4 Partitions consisting of odd distinct parts are not A_n -sign partitions

Let γ consists of odd distinct parts and let λ be the self conjugate partition with diagonal hook lengths equal to the parts of γ . Then

$$\chi_{\gamma,+}^{\lambda,\pm} = \frac{\epsilon \pm \sqrt{\epsilon \gamma_1 \dots \gamma_r}}{2}$$

with $\epsilon \in \{\pm 1\}$.

If $\epsilon = -1$ clearly $\chi_{\gamma,+}^{\lambda,\pm} \notin \{0, \pm 1\}$.

Assume now that $\epsilon = 1$. From γ being a partition of $n \geq 2$ it follows that $\gamma_1 \dots \gamma_r \geq 2$. In particular $\chi_{\gamma,+}^{\lambda,+} > 1$.

Similarly $\chi_{\gamma,-}^{\lambda,\pm}$ are not both in $\{0, \pm 1\}$ and then γ is not an A_n -sign partition. Together with Section 3 this prove that if γ is an A_n -sign partition then

$$\gamma \in (\text{Sign} \cup \overline{\text{Sign}}) \cap \{\text{even partitions not consisting of odd distinct parts}\}.$$

5 Elements of $\text{Sign} \cup \overline{\text{Sign}}$ are A_n -sign partitions

We will now prove that if

$$\gamma \in (\text{Sign} \cup \overline{\text{Sign}}) \cap \{\text{even partitions not consisting of odd distinct parts}\}$$

then γ is an A_n -sign partition. If $\gamma \in \text{Sign}$ this is easily proved in the next theorem.

Theorem 5.1. *Let $\gamma \in \text{Sign}$ be an even partition not consisting of odd distinct parts. Then γ is an A_n -sign partition.*

Proof. From Theorem 1.3 of [2] we have that γ is a sign partition. So $\chi_\gamma^\beta \in \{0, \pm 1\}$ for every $\beta \vdash n$. In particular $\chi_\gamma \in \{0, \pm 1/2, \pm 1\}$ and then also $\chi_\gamma \in \{0, \pm 1\}$ for every irreducible character χ of A_n , that is γ is an A_n -sign partition. \square

For $\gamma \in \overline{\text{Sign}}$ the proof is more complicated. It can be checked that $(1, 1, 1)$, $(2, 2)$, $(2, 2, 1)$, $(5, 4, 3, 2, 1)$, $(6, 5, 4, 1)$, $(6, 3, 2, 1, 1)$, $(9, 4, 3, 2, 1)$ and $(10, 5, 3, 2, 1)$ are A_n -sign partitions by looking at the corresponding characters tables. For the other elements of $\overline{\text{Sign}}$ we will use the following lemmas.

Lemma 5.2. *If $\gamma \in \overline{\text{Sign}}$ does not consist of odd distinct parts, then γ is an A_n -sign partition if and only if $\chi_\gamma^\beta \in \{0, \pm 1\}$ for every $\beta \vdash n$ not self conjugate with at least two γ_1 -hooks.*

Proof. Let $\gamma \in \overline{\text{Sign}}$. Then γ is an even partition. From $\gamma \in \overline{\text{Sign}}$ we have that $(\gamma_2, \dots, \gamma_r) \in \text{Sign}$ and that $|\gamma| \leq 3\gamma_1$. In particular any $\beta \vdash n$ has at most 3 γ_1 -hooks. As $(\gamma_2, \dots, \gamma_r) \in \text{Sign}$ and then it is a sign partition from Theorem 1.3 of [2], it follows that $\chi_\gamma^\beta \in \{0, \pm 1, \pm 2, \pm 3\}$ for every $\beta \vdash n$.

As γ does not consist of odd distinct parts, it is then enough to prove that $\chi_\gamma^\beta \in \{0, \pm 1\}$ for every $\beta \vdash n$ not self conjugate with at least two γ_1 -hooks, as then $\chi_\gamma \in \{0, \pm 1/2, \pm 1, \pm 3/2\}$, and so also $\chi_\gamma \in \{0, \pm 1\}$, for every irreducible character χ of A_n . \square

The next lemma is a generalization of Lemma 4.1 of [2].

Lemma 5.3. *Let $\gamma = (\gamma_1, \dots, \gamma_r)$ be a partition. Assume that $(\gamma_2, \dots, \gamma_r)$ is a sign partition and that $\gamma_2 + \dots + \gamma_r < 2a$. If β is a partition of n for which $\chi_\gamma^\beta \notin \{0, \pm 1\}$ then β has two γ_1 -hooks. Also if δ is obtained from β by removing a γ_1 -hook then $\chi_{(\gamma_2, \dots, \gamma_r)}^\delta \neq 0$. In particular each such δ has a γ_2 -hook.*

Proof. By assumption

$$n = |\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_r < 3a.$$

It follows that any partition of n has at most two γ_1 -hooks. As

$$\chi_\gamma^\beta = \sum_{(i,j): h_{i,j}^\beta = \gamma_1} \pm \chi_{(\gamma_2, \dots, \gamma_r)}^{\beta \setminus R_{i,j}^\beta}$$

and $(\gamma_2, \dots, \gamma_r)$ is a sign partition by Theorem 1.3 of [2], so that $\chi_{(\gamma_2, \dots, \gamma_r)}^{\beta \setminus R_{i,j}^\beta} \in \{0, \pm 1\}$ for each $(i, j) \in [\beta]$ with $h_{i,j}^\beta = \gamma_1$, the lemma follows. \square

We will now prove that all remaining elements of $\overline{\text{Sign}}$ are A_n -sign partitions if they do not consist of odd distinct parts.

Theorem 5.4. *If $a \geq 7$ then $(a, a-1, 4, 1)$ is an A_n -sign partition.*

Proof. As $|(a, a-1, 4, 1)| = 2a+4 < 3a$, so that any partition of $(a, a-1, 4, 1)$ has at most two a -hooks, we only need to consider, from Lemma 5.2, partitions of $2a+4$ with 2 a -hooks. Let β be any such partition. Then the a -core of β is a partition of 4 and so $\beta_{(a)} \in \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}$. As $\chi_\rho^{\lambda'} = \pm \chi_\rho^\lambda$ and $(\lambda')_{(q)} = (\lambda_{(q)})'$ for λ, ρ partitions and $q \geq 1$, we can assume that $\beta_{(a)} \in \{(4), (3, 1), (2, 2)\}$.

Assume first that $\beta_{(a)} = (4)$. From any a -core we can obtain exactly a different partitions by adding an a -hook to it. In this case they are given by

$$\{(a+4)\} \cup \{(4, i, 1^{a-i}) : 1 \leq i \leq 4\} \cup \{(a-i, 5, 1^{i-1}) : 1 \leq i \leq a-5\}.$$

If μ and ν are the partitions obtained from β by removing an a -hook, we can assume from Lemma 5.3 that

$$\mu, \nu \in \{(a+4), (4, 1^a), (5, 5, 1^{a-6}), (a-1, 5)\}$$

as the other partition do not have $(a-1)$ -hooks. From $a \geq 7$ and then

$$\chi_{(a-1,4,1)}^{(4,1^a)}, \chi_{(a-1,4,1)}^{(a-1,5)} = \pm \chi_{(4,1)}^{(4,1)} = 0$$

we can further assume that $\mu, \nu \in \{(a+4), (5, 5, 1^{a-6})\}$. Since $\mu \neq \nu$ and as we can recover β from the a -cores and a -quotients of μ and ν (there exists a unique such partition β), we obtain that $\beta = (a+4, 6, 1^{a-6})$ and then that

$$\chi_{(a,a-1,4,1)}^\beta = -\chi_{(a-1,4,1)}^{(5,5,1^{a-6})} + (-1)^{a-6} \chi_{(a-1,4,1)}^{(a+4)} = -(-1)^{a-6} \chi_{(4,1)}^{(5)} + (-1)^{a-6} = 0.$$

Assume next that $\beta_{(a)} = (3, 1)$. In this case

$$\begin{aligned} \mu, \nu \in & \{(a+3, 1), (a, 4), (3, 3, 2, 1^{a-4}), (3, 2, 2, 1^{a-3}), (3, 1^{a+1})\} \\ & \cup \{(a-i, 4, 2, 1^{i-2}) : 2 \leq i \leq a-4\}. \end{aligned}$$

Again from Lemma 5.3 we can assume that μ and ν have an $(a-1)$ -hook and so

$$\mu, \nu \in \{(a+3, 1), (a, 4), (3, 2, 2, 1^{a-3}), (3, 1^{a+1}), (4, 4, 2, 1^{a-6}), (a-2, 4, 2)\}.$$

Since

$$\begin{aligned} \chi_{(a-1,4,1)}^{(a+3,1)}, \chi_{(a-1,4,1)}^{(3,1^{a+1})}, \chi_{(a-1,4,1)}^{(4,4,2,1^{a-6})} &= \pm \chi_{(4,1)}^{(4,1)} = 0, \\ \chi_{(a-1,4,1)}^{(a-2,4,2)} &= \chi_{(4,1)}^{(3,1,1)} = 0 \end{aligned}$$

we can assume that $\mu, \nu \in \{(a, 4), (3, 2, 2, 1^{a-3})\}$, that is $\beta = (a, 4, 3, 1^{a-3})$. In this case

$$\chi_{(a,a-1,4,1)}^\beta = (-1)^{a-3} \chi_{(a-1,4,1)}^{(a,4)} + \chi_{(a-1,4,1)}^{(3,2,2,1^{a-3})} = (-1)^{a-2} \chi_{(4,1)}^{(3,2)} + (-1)^{a-3} \chi_{(4,1)}^{(3,2)} = 0.$$

Let now $\beta_{(a)} = (2, 2)$. Then

$$\begin{aligned} \mu, \nu \in & \{(a+2, 2), (a+1, 3), (2, 2, 2, 1^{a-2}), (2, 2, 1^a)\} \\ & \cup \{(a-i, 3, 3, 1^{i-2}) : 2 \leq i \leq a-3\}. \end{aligned}$$

From μ and ν having an $(a-1)$ -hook it follows

$$\mu, \nu \in \{(a+2, 2), (2, 2, 1^a), (a-2, 3, 3), (3, 3, 3, 1^{a-5})\}.$$

As $(a+2, 2)' = (2, 2, 1^a)$ and $(a-2, 3, 3)' = (3, 3, 3, 1^{a-5})$ and from β being self-adjoint if $\mu = \nu'$ (as the partitions obtained from β' by removing an a -hook are $\mu' = \nu$ and $\nu' = \mu$ in this case) we can assume that $\mu \in \{(a+2, 2), (2, 2, 1^a)\}$ and $\nu \in \{(a-2, 3, 3), (3, 3, 3, 1^{a-5})\}$. From $\chi_\rho^\lambda = \pm \chi_\rho^{\lambda'}$, we can further assume that $\mu = (a+2, 2)$. Then $\beta \in \{(a+2, a-1, 3), (a+2, 4, 3, 1^{a-5})\}$. From

$$\begin{aligned}\chi_{(a, a-1, 4, 1)}^{(a+2, a-1, 3)} &= -\chi_{(a-1, 4, 1)}^{(a-2, 3, 3)} - \chi_{(a-1, 4, 1)}^{(a+2, 2)} = -\chi_{(4, 1)}^{(2, 2, 1)} - \chi_{(4, 1)}^{(3, 2)} = 0, \\ \chi_{(a, a-1, 4, 1)}^{(a+2, 4, 3, 1^{a-5})} &= -\chi_{(a-1, 4, 1)}^{(3, 3, 1^{a-5})} + (-1)^{a-4} \chi_{(a-1, 4, 1)}^{(a+2, 2)} = (-1+1)(-1)^{a-4} \chi_{(4, 1)}^{(3, 2)} = 0\end{aligned}$$

it then follows that $(a, a-1, 4, 1)$ is an A_n -sign partition. \square

Theorem 5.5. *If $a \geq 8$ then $(a, a-3, 2, 1, 1)$ is an A_n -sign partition.*

Proof. Since $|(a, a-3, 2, 1, 1)| = 2a+1$ we only need to consider not self-adjoint $\beta \vdash 2a+1$ with two a -hooks from Lemma 5.2. In this case $\beta_{(a)} = (1)$. Let μ, ν be obtained from β by removing these a -hooks. Then

$$\mu, \nu \in \{(a+1), (1^{a+1})\} \cup \{(a-i, 2, 1^{i-1}) : 1 \leq i \leq a-2\}.$$

From Lemma 5.3 we can assume that μ and ν have an $(a-3)$ -hook, so that

$$\mu, \nu \in \{(a+1), (1^{a+1}), (a-1, 2), (a-3, 2, 1, 1), (4, 2, 1^{a-5}), (2^2, 1^{a-3})\}.$$

As $a-3 > 2$

$$\chi_{(a-3, 2, 1, 1)}^{(a-1, 2)}, \chi_{(2, 1, 1)}^{(2^2, 1^{a-3})} = \pm \chi_{(2, 1, 1)}^{(2, 2)} = 0$$

we can further assume that

$$\mu, \nu \in \{(a+1), (1^{a+1}), (a-3, 2, 1, 1), (4, 2, 1^{a-5})\}.$$

From $(a+1)' = (1^{a+1})$ and $(a-3, 2, 1^2)' = (4, 2, 1^{a-5})$ we can, as in the previous theorem, assume that $\mu = (a+1)$ and $\nu \in \{(a-3, 2, 1^2), (4, 2, 1^{a-5})\}$, so that $\beta \in \{(a+1, a-2, 1, 1), (a+1, 5, 1^{a-5})\}$. From $a > 5$ it follows

$$\begin{aligned}\chi_{(a, a-3, 2, 1, 1)}^{(a+1, a-2, 1, 1)} &= \chi_{(a-3, 2, 1, 1)}^{(a+1)} - \chi_{(a-3, 2, 1, 1)}^{(a-3, 2, 1, 1)} = 1 + \chi_{(2, 1, 1)}^{(1^4)} = 0, \\ \chi_{(a, a-3, 2, 1, 1)}^{(a+1, 5, 1^{a-5})} &= (-1)^{a-5} \chi_{(a-3, 2, 1, 1)}^{(a+1)} - \chi_{(a-3, 2, 1, 1)}^{(4, 2, 1^{a-5})} = (1-1)(-1)^{a-5} \chi_{(2, 1, 1)}^{(4)} = 0\end{aligned}$$

it then follows from Lemma 5.2 that $(a, a-3, 2, 1, 1)$ is an A_n -sign partition. \square

Theorem 5.6. *If $a \geq 12$ then $(a, a-5, 3, 2, 1)$ is an A_n -sign partition.*

Proof. As $|(a, a-5, 3, 2, 1)| = 2a+1$ we only need to consider not self-adjoint $\beta \vdash 2a+1$ with two a -hooks from Lemma 5.2. Also in this case $\beta_{(a)} = 1$, so that if μ and ν are obtained from β by removing an a -hook and they have an $(a-5)$ -hook (as we can apply Lemma 5.3), we have that

$$\mu, \nu \in \{(a+1), (1^{a+1}), (a-1, 2), (a-2, 2, 1), (a-3, 2, 1^2), (a-5, 2, 1^4), (6, 2, 1^{a-7}), (4, 2, 1^{a-5}), (3, 2, 1^{a-4}), (2^2, 1^{a-3})\}.$$

From $a-5 > 4$ and then

$$\begin{aligned} \chi_{(a-5, 3, 2, 1)}^{(a-1, 2)}, \chi_{(a-5, 3, 2, 1)}^{(4, 2, 1^{a-5})} &= \pm \chi_{(3, 2, 1)}^{(4, 2)} = 0, \\ \chi_{(a-5, 3, 2, 1)}^{(a-2, 2, 1)}, \chi_{(a-5, 3, 2, 1)}^{(3, 2, 1^{a-4})} &= \pm \chi_{(3, 2, 1)}^{(3, 2, 1)} = 0, \\ \chi_{(a-5, 2, 1^2)}^{(a-3, 2, 1^2)}, \chi_{(a-5, 3, 2, 1)}^{(2^2, 1^{a-3})} &= \pm \chi_{(3, 2, 1)}^{(2, 2, 1, 1)} = 0 \end{aligned}$$

we can assume that

$$\mu, \nu \in \{(a+1), (1^{a+1}), (a-5, 2, 1^4), (6, 2, 1^{a-7})\}$$

and then again that $\mu = (a+1)$ and $\nu \in \{(a-5, 2, 1^4), (6, 2, 1^{a-7})\}$, so $\beta \in \{(a+1, a-4, 1^4), (a+1, 7, 1^{a-7})\}$. From $a > 11$ we have that

$$\begin{aligned} \chi_{(a, a-5, 3, 2, 1)}^{(a+1, a-4, 1^4)} &= \chi_{(a-5, 3, 2, 1)}^{(a+1)} - \chi_{(a-5, 3, 2, 1)}^{(a-5, 2, 1^4)} = 1 + \chi_{(3, 2, 1)}^{(1^6)} = 0, \\ \chi_{(a, a-5, 3, 2, 1)}^{(a+1, 7, 1^{a-7})} &= (-1)^{a-7} \chi_{(a-5, 3, 2, 1)}^{(a+1)} - \chi_{(a-5, 3, 2, 1)}^{(6, 2, 1^{a-7})} = (1-1)(-1)^{a-7} \chi_{(3, 2, 1)}^{(6)} = 0. \end{aligned}$$

In particular $(a, a-5, 3, 2, 1)$ is an A_n -sign partition. \square

Theorem 5.7. *If $b+1 \leq a \leq 2b-2$ then $(a, b, a-b+1)$ is an A_n -sign partition if it does not consists of odd distinct parts.*

Proof. From $b+1 \leq a \leq 2b-2$ it follows that $a > b > a-b+1 > 1$. Again from Lemma 5.2 we only need to consider partitions β of $|(a, b, a-b+1)| = 2a+1$ with two a -hooks which are not self-adjoint. From Lemma 5.3 in this case μ and ν have a b -hook, so that

$$\mu, \nu \in \{(a+1), (1^{a+1})\} \cup \{(a-i, 2, 1^{i-1}) : 1 \leq i \leq a-b-2, i = a-b, i = b-1 \text{ or } b+1 \leq i \leq a-2\}.$$

From $b > a-b+1$ it follows that any partition of $a+1$ has at most one b -hook, so that

$$\chi_{(b, a-b+1)}^{(a-i, 2, 1^{i-1})} = \chi_{(a-b+1)}^{(a-b-i, 2, 1^{i-1})} = 0$$

for $1 \leq i \leq a - b - 2$ and similarly for $b + 1 \leq i \leq a - 2$. As in the previous theorems we can assume that $\mu = (a + 1)$ and $\nu \in \{(b, 2, 1^{a-b-1}), (a - b + 1, 2, 1^{b-2})\}$, that is $\beta \in \{(a + 1, b + 1, 1^{a-b-1}), (a + 1, a - b + 2, 1^{b-2})\}$. As

$$\begin{aligned}\chi_{(a,b,a-b+1)}^{(a+1,b+1,1^{a-b-1})} &= (-1)^{a-b-1} \chi_{(b,a-b+1)}^{(a+1)} - \chi_{(b,a-b+1)}^{(b,2,1^{a-b-1})} = (-1)^{a-b-1} + \chi_{(a-b+1)}^{(1^{a-b+1})} = 0, \\ \chi_{(a,b,a-b+1)}^{(a+1,a-b+2,1^{b-2})} &= (-1)^{b-2} \chi_{(b,a-b+1)}^{(a+1)} - \chi_{(b,a-b+1)}^{(a-b+1,2,1^{b-2})} = (-1)^{b-2} (1 - \chi_{(a-b+1)}^{(a-b+1)}) = 0\end{aligned}$$

we have that $(a, b, a - b + 1)$ is an A_n -sign partition if it does not consists of odd distinct parts and $b + 1 \leq a \leq 2b - 2$. \square

Theorem 5.8. *If $b+2 \leq a \leq 2b$ then $(a, b, a-b-1, 1)$ is an A_n -sign partition if it does not consists of odd distinct parts.*

Proof. Here $a > b > a - b - 1 \geq 1$, as $b + 2 \leq a \leq 2b$. From Lemma 5.2 we only need to compute $\chi_{(a,b,a-b-1,1)}^\beta$ for $\beta \vdash 2a$ not self-adjoint with two a -hook. If μ and ν are obtained from such a β by removing an a -hook then μ and ν are a -hooks. From $a \leq 2b$ it follows that μ and ν have at most one b -hook. If such a b -hook exists then

$$\mu, \nu \in \{(a - i, 1^i) : 0 \leq i \leq a - b - 1 \text{ or } b \leq i \leq a - 1\}.$$

For $1 \leq i \leq a - b - 2$

$$\chi_{(b,a-b-1,1)}^{(a-i,1^i)} = \chi_{(a-b-1,1)}^{(a-i-b,1^i)} = 0$$

and similarly for $b + 1 \leq i \leq a - 2$. So from Lemma 5.3 we can assume that

$$\mu, \nu \in \{(a), (b + 1, 1^{a-b-1}), (a - b, 1^b), (1^a)\}.$$

Also here we can assume $\mu = (a)$ and $\nu \in \{(b + 1, 1^{a-b-1}), (a - b, 1^b)\}$, that is $\beta \in \{(a, b + 2, 1^{a-b-2}), (a, a - b + 1, 1^{b-1})\}$. From

$$\begin{aligned}\chi_{(a,b,a-b-1,1)}^{(a,b+2,1^{a-b-2})} &= (-1)^{a-b-2} \chi_{(b,a-b-1,1)}^{(a)} - \chi_{(b,a-b-1,1)}^{(b+1,1^{a-b-1})} = (-1)^{a-b} - \chi_{(a-b-1,1)}^{(1^{a-b})} = 0, \\ \chi_{(a,b,a-b-1,1)}^{(a,a-b+1,1^{b-1})} &= (-1)^{b-1} \chi_{(b,a-b-1,1)}^{(a)} - \chi_{(b,a-b-1,1)}^{(a-b,1^b)} = (-1)^{b-1} (1 - \chi_{(a-b-1,1)}^{(a-b)}) = 0\end{aligned}$$

it follows that $(a, b, a - b - 1, 1)$ is an A_n -sign partition if it does not consists of odd distinct parts. \square

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